Mathematical modelling and thermal stress intensity factors evaluation for an interface crack in the presence of a system of cracks in functionally graded/homogeneous bimaterials

Vera Petrova\textsuperscript{a,b,*}, Siegfried Schmauder\textsuperscript{b}

\textsuperscript{a} Department of Mathematics, Voronezh State University, University Sq. 1, Voronezh 394006, Russia
\textsuperscript{b} IMWF, University of Stuttgart, Pfaffenwaldring 32, D-70569 Stuttgart, Germany

\textbf{A R T I C L E  I N F O}

Article history:
Received 26 October 2010
Received in revised form 14 February 2011
Accepted 20 February 2011
Available online 22 March 2011

Keywords:
Functionally graded materials
Interface
Cracks
Crack closure
Thermal fracture
Thermal stress intensity factor

\textbf{A B S T R A C T}

The work is devoted to mathematical modeling of the fracture processes in the vicinity of an interface crack in functionally graded/homogeneous bimaterials with internal defects subjected to a thermal flux. A previously obtained solution (Petrova and Schmauder, 2009, 2011)\textsuperscript{[7,8]} is used and supplemented with the additional possibility to take into account the crack closure. The solution is based on the integral equation method and it is assumed that thermal properties of the functionally graded material (FGM) have exponential form. For a special case where an interface crack length is much larger than the internal cracks in the FGM an asymptotic analytical solution of the problem is obtained as series of a small parameter (the ratio between sizes of the internal and interface cracks). Analyses of the effects of the location and orientation of the cracks, the material non-homogeneity parameters and the crack closure effects on the thermal stress intensity factors of the interface crack in FGM/homogeneous bimaterials are performed. Examples of some FGM/homogeneous bimaterial combinations (i.e., metal/metal, ceramic/metal) are discussed.

\textcopyright 2011 Elsevier B.V. All rights reserved.

1. Introduction

Bimaterials and functionally graded materials (FGMs) are widely used in engineering structures, which are subject to different loads: mechanical, thermal and combinations of them. FGMs are designed so that to decrease bimaterial mismatch and residual stresses at the interface and prevent debonding along the interface. However, the interaction of defects causes additional stresses near interfaces and can lead to enhance or suppress crack propagation, so that crack interaction problems in FGMs and bimaterials are important for the investigation of fracture strength of materials.

Great progress was achieved in computational modelling of FGMs and cracks in FGMs (see a review\textsuperscript{[1]}). Finite element method (FEM) modelling, boundary integral analysis and their different modifications are widely used for these purposes\textsuperscript{[2]}. Boundary integral methods have advantages because of only boundary discretization is need in comparison with volume discretization in FEMs. Many integral equation methods, however, are based on knowing fundamental solutions of the corresponding partial differential equations with variable coefficients and have therefore limitations. One of the fundamental solutions, Green’s functions are known only for the simplest cases, for example, the Green’s function for an FGM wherein the material properties vary exponentially through the solid has been obtained in closed form for heat conduction problems and elasticity problems\textsuperscript{[3]}, but it is difficult to use this solution for crack interaction problems in FGMs. Most of these methods allow studying complicated boundary value problems for FGMs, but they require large efforts to develop special programs or using commercial programs and it consumes much computational time.

Another way is to derive approximate analytical solutions. It is evident, that these solutions can be obtained for special cases and with some assumptions and, hence, they have limitations. Meanwhile, these solutions are necessary for a first estimation of the mechanical and thermal parameters of FGMs with cracks, for better understanding the processes and possibly to show the direction of further investigations by numerical methods. Besides, the analytical solutions can check correctness of numerical solutions.

Crack interaction problems in homogeneous materials have been extensively investigated and large number of solutions have been obtained for different crack system configurations and different thermal and mechanical loadings\textsuperscript{[4]}. Many papers also devoted to different models and semi-analytical solutions for cracks in FGMs, but crack interaction problems were obtained only for special crack arrangements (see, for example\textsuperscript{[5,6]}).

E-mail addresses: veraep@gmail.com (V. Petrova), Siegfried.Schmauder@imw.f.uni-stuttgart.de (S. Schmauder).

Available online 22 March 2011
Accepted 20 February 2011
Received in revised form 14 February 2011
Received 26 October 2010

The present work is devoted to modelling of the fracture processes in the vicinity of an interface crack in functionally graded/homogeneous bimaterials with internal defects subjected to a thermal flux applied at infinity. It is assumed that thermal properties of FGMs possess exponential form. In previous works [7,8] an approximate analytical solution was obtained for a special case where the interface crack length is larger than nearby internal cracks in the FGM. The thermal stress intensity factors (TSIFs) for the interface crack were derived as series of a small parameter (the ratio between sizes of the internal and interface cracks). Parametric analyses of the effects of the location and orientation of the cracks and the material non-homogeneity parameters on the thermal stress intensity factors of an interface crack in FGM/homogeneous bimaterials were performed in [8]. This solution is used in the present investigation and supplemented with the additional possibility to take into account the crack closure.

2. Formulation of the problem

2.1. Geometry of the problem and assumptions

The geometry of the problem is shown in Fig. 1. A bimaterial is composed of a functionally graded material (denoted by number 1) and a homogeneous material (denoted by number 2). The bimaterial is perfectly bonded with exception of an interface crack of length 2a0. It is assumed that the FGM contains N cracks of length 2a0. Cartesian coordinates (x, y) are centered at the midpoint of the interface crack; the x-axis lies along the interface line. Local coordinate systems (xk, yk) are attached to each internal crack and to the midpoint coordinate (x0, y0) and an inclination angle θk to the interface, i.e. to the x-axis (Fig. 1).

The bimaterial is subjected to a heat flux of intensity q applied at infinity. The cracks are thermally isolated and traction free.

It is assumed that properties of the functionally graded material depend only on the coordinate y. The thermal conductivity coefficient and the thermal expansion coefficient are

\[ k_1(y) = k_0 e^{\delta y}, \quad \alpha_1 = a_0 e^{\omega y} \]

where the constant \( k_0 \) is the thermal conductivity and \( a_0 \) is the thermal expansion coefficient of the interface and of material 2, while \( \delta \) and \( \omega \) are inhomogeneity parameters of the FGM. The Young’s modulus and Poisson’s ratio are assumed to be constant, \( E_j = const, \)

\[ \nu_j = \text{const} \ (j = 1, 2). \]

Thus, the material is elastically homogeneous, but thermally non-homogeneous.

The relation between global coordinates (x, y) and the local coordinate systems (xk, yk) can be written in complex form as follows:

\[ z = z_1 + z_2 e^{\theta k}, \quad \delta_1 = \delta \sin \theta_k, \quad \delta_2 = \delta \cos \theta_k, \]

\[ \alpha_1 = \alpha \sin \theta_k, \quad \alpha_2 = \alpha \cos \theta_k. \]

(2)

The uncoupled, quasi-static thermoelastic theory is applicable to this problem that is the temperature distribution is independent of the mechanical field, and the solution consists of the determination of the temperature field and the determination of the thermal stresses.

2.2. Thermal problem

For the solution of the problem a superposition principle is used. Due to this principle the temperature field \( T_j(y) \) in the bimaterial with cracks is presented as \( T_j(x, y) = T_j^0(x, y) + T_j(x, y) \) – the temperature distribution in a bimaterial in the absence of cracks, \( T_j(x, y) \) – the temperature perturbation caused by the cracks.

The thermal boundary conditions for thermally isolated cracks and continuity conditions for the temperature perturbation \( T_j(x, y) \) read as follows:

\[ k_1 \frac{\partial T_1(x, +0)}{\partial y} = k_2 \frac{\partial T_2(x, -0)}{\partial y} = q_0(x)||x| < a_0), \]

\[ k_1 \frac{\partial T_1(x, +0)}{\partial y} = q_n(x)||x| < a_n), \]

\[ k_1 \frac{\partial T_1(x, +0)}{\partial y} = k_2 \frac{\partial T_2(x, -0)}{\partial y}, \quad T_1(x, +0) = T_2(x, -0)|x| > a_0, \quad y = 0. \]

\[ T_1(\pm a_0, +0) = T_2(\pm a_0, -0), \quad T_{m}(\pm a_n, +0) = T_{m}(\pm a_n, -0), \]

and the temperature perturbation vanishes at infinity. Here

\[ q_0 = -k_1 \left( \frac{\partial T_j}{\partial y} \right)_{y=0}, \quad q_n = -k_1 \left( \frac{\partial T_j}{\partial y} \right)_{y=\theta_n}. \]

The signs ‘+’ and ‘-’ denote the limiting values of the functions on the upper and lower surfaces of the crack or the interface, respectively.

The heat conduction equation for the steady state temperature in FGMs with thermal conductivity coefficient Eq. (1) is given by

\[ \nabla^2 T_j + \delta \frac{\partial T_j}{\partial y} = 0, \]

and for material 2 with \( \delta = 0 \) we have the Laplace equation \( \nabla^2 T_2 = 0 \), where \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \). Solving the temperature problem for an undamaged FGM/homogeneous bimaterial (Eq. (6) for \( T_j^0 \)) the thermal fluxes on the crack lines, Eq. (5), are obtained as

\[ q_0 = -k_1 q, \]

\[ q_n = -k_1 q \exp(-\theta_n^2) \exp(-\theta_n \sin \theta_k) \cos \theta_k \ (n = 1, 2, \ldots, N). \]

(7)

Supposing that the non-homogeneity of the FGM is revealed only in non-homogeneous thermal fluxes (Eq. (7)) on crack surfaces, the perturbation problem can be solved by the method presented in [4] using similar integral equations. The system of \( N + 1 \) singular integral equations for the unknown functions \( \gamma_k \) is written as
\[
\int_{-a}^{a} \frac{\gamma_{k}(t)}{t-x} dt + \sum_{k=0}^{N} \int_{-a}^{a} \frac{\gamma_{k}'(t)P_{nk}(t,x)}{t-x} dt = \pi q_{n}(x), \quad |x| < a_{n}, \quad (8)
\]

\[
\int_{-a}^{a} \gamma_{k}'(t) dt = 0, \quad n = 0, 1, \ldots, N. \quad (9)
\]

Eq. (9) is the condition of the temperature continuity at the crack endpoints and represents Eq. (4) rewritten in the integral form. The functions \(q_{n}(x)\) at the right part of Eq. (8) are determined by Eq. (7). The functions \(\gamma_{k}'\) are the derivatives of temperature jumps on the crack lines

\[
2\gamma_{k}(t) = T_{k}^{+} - T_{k}^{-}.
\]

The regular kernels \(P_{nk}\) \((n, k = 0, 1, \ldots, N)\) contain the geometry of the problem and are given by the expressions:

\[
P_{nk} = \text{Re}\left[\frac{e^{ikx} (e^{ikx} - \lambda e^{ikx} - \lambda^{2} e^{ikx})}{(1 + \lambda^2) (e^{ikx} - \lambda e^{ikx} - \lambda^{2} e^{ikx})}\right].
\]

In Eq. (11) and in all expressions below \(\text{Re}\) denotes the real part of \(P_{nk}\). The condition of temperature continuity at the crack endpoints is \(T_{k}^{+} - T_{k}^{-} = 0\). The condition at infinity is \(\sigma_{yy} - \gamma_{yy}^{+} - \gamma_{yy}^{-} = 0, \ |x| = a_{n}, \ y = 0\).

\[
(\sigma_{yy} - \gamma_{yy}^{+}) = (\sigma_{yy} - \gamma_{yy}^{-}) = 0, \quad |x| < a_{n}, \quad y = 0.
\]

The continuity conditions at the interface are satisfied, i.e. the stresses are equal and displacements are equal. The condition at infinity is \(\sigma_{yy} = 0, x^{2} + y^{2} \to \infty\). Here \(\tau_{yy}\) and \(\sigma_{yy}\) are normal and shear stresses, \(n_{i}\) and \(u_{i}\) are normal and shear displacements, \(j = 1, 2\) for materials 1 and 2.

Because of we suppose that the material is elastically homogeneous we can use directly the method presented in [4]. The system of singular integral equations for the traction-free cracks is written as [9]:

\[
\int_{-a}^{a} G_{0}(t) \frac{dt}{t-x} + \sum_{k=0}^{N} \int_{-a}^{a} \left[ G_{0}(t) K_{nk}(t,x) + \frac{\partial}{\partial x} \left[ G_{0}(t) L_{nk}(t,x) \right] \right] dt = 0, \quad |x| < a_{n},
\]

\[
\int_{-a}^{a} G_{0}(t) dt = iA_{k} = -2i \int_{-a}^{a} \tilde{b}_{k} t \gamma_{k}^{0}(t) dt, \quad n = 0, 1, \ldots, N. \quad (13)
\]

The complex conjugate. The function \(G_{0}\) consists of two parts as

\[
G_{0}(t) = g_{0}(t) + 2i \tilde{b}_{k} \gamma_{k}^{0}(t),
\]

where functions \(\gamma_{k}^{0}\) were found from the thermo-conductivity problem, while unknown functions \(g_{0}(x)\) are the derivatives of the displacement jumps on the crack lines

\[
g_{0}'(x) = \frac{2(\mu + (1+\kappa)/\kappa)(u_{n}^{0} - u_{n}^{-}) + i(\nu_{n}^{0} - \nu_{n}^{-})}{\partial x}.
\]

In Eqs. (13)–(15) \(\tilde{b}_{k}\) is \(\tilde{b}_{k} = \tilde{b}_{k} E^{i} E^{i}\). The following applies to the plane strain case \(E^{i} = E/(1+\kappa)\) and \(\kappa = 3 - 4\nu\) while the plane stress condition corresponds to \(E^{i} = E/(1+\kappa)/\kappa\) and \(\kappa = (3 - 4\nu)\). \(\mu = E/2(1+\nu)\) is the shear modulus. Taking into account Eq. (2) the parameter \(\tilde{b}_{k}\) is written as

\[
\tilde{b}_{k} = \tilde{b}_{k} E^{i} = \tilde{b}_{k} E^{i} \epsilon^{(a_{n}/2a_{n})} \epsilon^{(a_{n}/2a_{n})}, \quad \tilde{b}_{k} = \tilde{b}_{k} E^{i}.
\]

The condition of Eq. (14) provides that displacements are single-valued at the end points of the cracks. The condition of temperature continuity at the crack tips \(\gamma_{0}^{0}(\pm a_{n}) = 0\) is also taken into account in Eq. (14). The regular kernels \(K_{nk}\) and \(L_{nk}\) contain the geometry of the problem and can be found in [4,8] or [9].

### 3. Solution of the problem

The solution is derived for a special case when an interface crack is significantly larger in size than internal cracks in the FGM. The asymptotic analytical solution of the problem is obtained as a series of the small parameter which is equal to the ratio of the size of small internal cracks to the interface crack size. The method was first suggested by Romalis and Tamuzs at 1984 and then was used for different macro–microcrack interaction problems for homogeneous materials [4]. In the interest of brevity the solution will not be repeated here in details. For the macro–microcrack interactions in a homogeneous material under thermal flux the full solution can be found in [4].

#### 3.1. Solution by small parameter method

Let us assume that all internal cracks in the FGM have the same size \(2a_{n} = 2a\) \(\left(k = 1, 2, \ldots, N\right)\), for example, they have the characteristic size of a grain size of the material. Suppose also that \(2a \ll 2a_{0}\). In this case the small parameter is \(\tilde{b}_{k} = a_{n}/a_{0} \approx 1\). Introducing non-dimensional coordinates \(\chi\) and \(\tau = x/a_{0}\chi\) and \(\tau = a_{0}/\tau\), Eq. (8) are rewritten in dimensionless variables. The unknown functions \(\gamma_{k}^{0}\) (10) are sought as a power series with respect to \(\tilde{b}_{k}\)

\[
\gamma_{k}^{0} = \sum_{p=0}^{\infty} \gamma_{p}^{0}(\chi) \tilde{b}_{k}^{2p}, \quad n = 0, 1, \ldots, N.
\]

The regular kernels \(P_{nk}\) (11) are expanded in series of \(\tilde{b}_{k}\), too. For convergence of these series the condition \(\chi/\chi^2 < 1\) must be satisfied. It is fulfilled if cracks are not intersected.

By substituting the series (18) and kernels into Eq. (8), and equating the coefficients of corresponding powers of \(\tilde{b}_{k}\), a recurrent system of equations for coefficients \(\gamma_{p}^{0}\) in series (18) is formed (see [4] for details). After solving the recurrent equations, the solution can be represented as a Maclaurian series over even powers of the small parameter \(\tilde{b}_{k}\) (all coefficients with odd second index are equal to zero).

The second approximation for the temperature jump \(\gamma_{0}^{0}\) on the interface crack line is derived in closed form as

\[
\gamma_{0}^{0} = \gamma_{00}^{0}(\chi) + \tilde{b}_{k}^{2} \gamma_{02}^{0}(\chi).
\]

The full expression of this function can be found in [8]. The zeroth approximation \(\gamma_{0}^{0}\) in Eq. (19) corresponds to an isolated interface crack and the second \(\gamma_{02}^{0}\) is taking into account the influence of each microcrack on the interface crack.

The solution (19) is used in the thermoelastic problem. The scheme of the solution of the thermoelastic problem is the same as for the solution of the thermal problem. The second approximation for the function \(G_{0}\) is given as [7]

\[
G_{0}(\chi) = G_{00}(\chi) + G_{02}(\chi) \tilde{b}_{k}^{2}.
\]

Using this function the thermal stress intensity factors are obtained.

#### 3.2. Stress intensity factors

In this work we consider elastically homogeneous materials so that we can use the classical definition of the stress intensity factor. It should be noted, that the crack tip singular field in FGMs has the same form as in homogeneous media [10] and the concept of the stress intensity factors can be also applied directly to cracks in FGMs. Besides, the interface crack between the FGM and the homogeneous material with smooth transition between these materials is also classical crack with square-root singularities at the crack tips. Therefore, in all these cases the thermal stress intensity factors (TSIFs) are found by [4,9]

\[
k_{s}^{1} - ik_{s}^{2} = \lim_{x \to \chi} \sqrt{a_{0}(1 - \chi^2)} G_{0}(\chi), \quad (n = 0, 1, 2, \ldots, N),
\]
where the upper part of the “±” or “/C7” signs refers to the right tip and the lower part to the left tip of cracks. Substituting Eq. (20) into Eq. (21), the thermal stress intensity factors at the interface crack tips are obtained up to a2 as

\[ k_{m0}^2 = \frac{e^{i\phi_k}q_k\alpha_0\sqrt{a_0}}{\sqrt{2}} \sum_{k=1}^{n} \{ \text{Re}(f_{m0}^k)\text{Im}(m_{k1} - n_{k1}) + \text{Im}(f_{m0}^k)\text{Re}(m_{k1} + n_{k1}) + 2\exp(\alpha_0\text{Im}(w_k))f_j^k(\delta)\text{Im}(m_{k0} - n_{k0}) \} \].

(22)

The interaction of cracks leads to mixed mode conditions in the interface crack surfaces. The thermal macro–microcrack interaction problem in homogeneous materials taking into account the crack closure was studied in Ref. [4]. We can apply this scheme for the solution in the present investigation.

The asymptotic solution to the problem of the interaction between an interface crack and internal microcracks in the FGM under the thermal loading is obtained with the following assumptions [4]:

- a solitary contact free portion of the crack with the unknown length 2c is located on the open part of the interface macrocrack, a local coordinate system (x0, y0) is attached to this part of the interface crack (Fig. 2);
- the temperature distribution does not depend on the crack closure. It means that on the closed portion of the crack the heat flux remains negligible;
- a smooth contact is assumed on the closed portion of the crack.

Due to the second assumption, the temperature distribution is the same as was obtained in previous section. Therefore, only the problem of thermoelasticity needs to be considered. To account for the crack closure, it is necessary to reformulate boundary conditions. For completely open cracks, the boundary conditions (12) are valid. On a closed portion of the interface crack or on the closed edges of microcracks, the shear stresses τzk and the transverse displacement jumps \[ \{n_{k0}\} \] are zero:

\[ \tau_z^k(x, 0) = 0, |n_{k0}| = 0. \]

(27)

For the case of open cracks, the unknowns are the shear and transverse displacement jumps \[ \{n_{k0}\} \] and \[ \{n_{k1}\} \] Eq. (16). For closed cracks, the shear displacement jumps and normal tractions are unknown. The boundary value problem consisting of the system of Eq. (13) with boundary conditions (12) and (27) decouples into two problems: one for the real part and another for the imaginary one. Separating the real and imaginary parts of the system of equations are obtained [4]. The second approximation of the TSIFs at the interface crack tips are obtained as

\[ \text{Im}(f_{m0}^k) = \sum_{k=1}^{n} \{ \text{Re}(f_{m0}^k)\text{Im}(m_{k1} - n_{k1}) + \text{Im}(f_{m0}^k)\text{Re}(m_{k1} + n_{k1}) + 2\exp(\alpha_0\text{Im}(w_k))f_j^k(\delta)\text{Im}(m_{k0} - n_{k0}) \} \].

(28)

and the TSIFs at the microcracks tips are

\[ k_{m0}^2 = \frac{e^{i\phi_k}q_k\alpha_0\sqrt{a_0}}{\sqrt{2}} \sum_{k=1}^{n} \{ \text{Re}(f_{m0}^k)\text{Im}(m_{k1} - n_{k1}) + \text{Im}(f_{m0}^k)\text{Re}(m_{k1} + n_{k1}) + 2\exp(\alpha_0\text{Im}(w_k))f_j^k(\delta)\text{Im}(m_{k0} - n_{k0}) \} \].

(29)

The expression for \( k_{m0} \) is defined by Eq. (23) in which the term with \( \text{Im}(f_{m0}^k) \) should be multiplied by \( V \).

In Eq. (28) \( m_{k0}, n_{k0}, m_{k1}, \) and \( n_{k1} \) are obtained from Eq. (25) by setting \( (w_k - d_k)/\epsilon \) instead of \( w_k \). The parameters \( \epsilon = c/a_0 \) and \( d_0 = a_0 z_c \) is the center of the open portion of the interface crack (Fig. 2).

By virtue of the boundary condition (27) the parameter \( V \) is introduced to Eq. (28) as

\[ V = \begin{cases} 0 & \text{if } k_{m0} < 0 \\ 1 & \text{if } k_{m0} > 0 \end{cases} \]

(30)

which allows to take into account the microcrack closure. \( k_{m0} \) is determined by Eq. (29).

The function of TSIF \( k_{m0} \) contains the unknown parameters of the open portion of the interface crack. The crack is closed smoothly and unknown coordinates of the points of separation of

![Fig. 2. An interface crack with a contact zone and microcracks.](image-url)
the closed and open parts of the interface crack are found from the condition that the singularity vanishes at these points (see [4,11])

\[ k_0(z+c) = 0. \]  

(31)

This condition is written in the local system \((x_0, y_0)\), Fig. 2.

An iterative procedure is used for finding the length of non-contacting regions of the interface crack. The first approximation of the length of the overlapping portion is replaced by every next approximation until the equality (31) is satisfied with sufficient accuracy. The same crack closure algorithm was also applied in [12] to treat the problem of having negative \(k_1\) for a partially insulated interface crack between a functionally graded coating and a homogeneous substrate subjected to both thermal and mechanical loading.

In the frame of this approximation it has been established that small cracks are existed in only two modes – fully open or full closed. Microcracks depending on their location and orientation can cause either full or partial closure of the interface crack. The value of \(k_0\) at the interface macrocrack tip can be evaluated disregarding the closed portion of the interface crack, however, for a correct determination of \(k_1\) the presence of the closed portion should be taken into account.

5. Results and discussion

The obtained asymptotic analytical formulae for TSIFs Eqs. 22, 23, 28, and 29 allow investigating the influence of different arrays of microcracks and parameters of inhomogeneity of FGMs on the interface crack. Assume that all microcracks have the same angle of inclination \(\theta\) to the \(x\)-axis. The microcrack centers are presented by \(x_0 = a_0\cos\theta, y_0 = a_0\sin\theta\) \((n, m = ±1, ±2, \ldots)\), with \(r = s = 5\), \(w_0 = (x_0 + y_0)/a_0\) (Fig. 3). The schemes of different arrangements of multiple microcracks in the FGM are shown in Fig. 3. TSIFs \(k_{0\theta}\) are normalized by \(|k_{0\theta}|\) [26] and denoted by \(K_1\) and \(K_2\) in the figures. The calculations were performed with \(\lambda = 0.1\). The non-dimensional inhomogeneity parameters of thermal conductivity and of thermal expansion are \(a_{0\phi}\) and \(a_{0\alpha}\), but in the figures we will leave the designation \(\phi\) and \(\alpha\). Besides, we can put \(a_0\) equals to 1 without reducing generality.

The values of the inhomogeneity parameters are estimated on the following reasons. From exponential form of the thermal conductivity Eq. (1) the inhomogeneity parameter \(a\) is \(a = (1/y^*)\ln(k_1/k_2)\). That is, the value of \(a\) depends on the ratio of material properties and the value of \(y^*\) the region where these properties of the FGM vary. We consider infinite domain and so that we suppose that the value of \(a\) is changed slowly, we take \(-1.0 < \delta < 1.0\). Besides, if \(k_1 > k_2\) then \(\delta > 0\), and if \(k_1 < k_2 – \delta < 0\). The same concerns the inhomogeneity parameter of the thermal expansion coefficient \(\alpha\).

At first we will use the formulae (22)–(25). Normalized TSIFs \(K_1\) and \(K_2\) as functions of inclination angle \(\theta\) of a microcrack with midpoint coordinate \(w_0 = 1.2 + 0.2\) (Fig. 3a) are shown in Fig. 4 for different values of the non-homogeneity parameter \(\delta\): \(\delta = 0\) – large dashed line, \(0.5\) – small dashed line, \(1\) – solid line. The curves of TSIFs at the right interface crack tip are denoted by \(K_{1\theta}\) and at the left tip by \(K_{1\alpha}\). The figures indicate that the TSIFs at the right interface crack tip are higher than at the left tip. The \(\delta\) is slightly influencing on the TSIFs and if the microcrack inclination angle approaches \(\pi/2\) this influence disappears.

Fig. 5 shows the influence of the non-homogeneity parameter \(a\) of thermo-conductivity and the microcrack inclination angle \(\theta\) on the thermal stress intensity factors \(K_1\) and \(K_2\) at the interface crack tips for the uniform distributed system of microcracks (Fig. 3c). We can see that \(K_1\) at the right interface crack tip is greater than at the left tip for all inclination angles \(\theta\) of microcracks and for all \(a\) values.

The presented in Figs. 4 and 5 results correspond to the case when the materials have similar thermal expansion coefficients, that is \(\alpha = 0\), and the thermal conductivity \(k\) is increased with increasing the coordinate \(y\), that is \(\delta > 0\). It could be, for example, for an FGM/homogeneous bimaterial (MoSi\(_2\)/Al\(_2\)O\(_3\))/Al\(_2\)O\(_3\) that is in the lower part in Fig. 1 the material is alumina Al\(_2\)O\(_3\) with \(k^\text{Al}_2\text{O}_3 = 25\, \text{Wm}^{-1}\text{K}^{-1}\) and in the upper part the material is an FGM MoSi\(_2\)/Al\(_2\)O\(_3\) gradually varying from molybdenum disilicide MoSi\(_2\) with \(k^\text{MoSi}_2 = 52\, \text{Wm}^{-1}\text{K}^{-1}\) to Al\(_2\)O\(_3\). At the same time MoSi\(_2\) and Al\(_2\)O\(_3\) have similar thermal expansion coefficients [13,14].

TSIFs \(K_1\) and \(K_2\) at the right interface crack tip as functions of the inhomogeneity parameter \(\omega\) of thermal expansion coefficient and for different \(\delta\) is presented in the Fig. 6 for the non-symmetrically disposed microcrack system ahead of the interface crack (Fig. 3b) for \(\theta = 0\) and in the Fig. 7 for \(\theta = \pi/4\). The same system of cracks but with different inclination angles \(\theta\) produces different influence on the TSIFs. The microcracks with \(\theta = 0\) cause the interface crack closure – \(K_1\) is negative for most parameters of \(\delta\) and \(\omega\). At the same time for \(\theta = \pi/4\) \(K_1\) is positive for all inhomogeneity parameters. The examples of the material combinations are the following: para-

![Fig. 3. Schemes of locations of the interface crack and microcracks: (a) one crack with midpoint coordinate \(w_0 = 1.2 + 0.2\); (b) non-symmetrically disposed microcrack system ahead of the interface crack; (c) a symmetric system of microcracks above the interface and (d) randomly orientated system of cracks.](image-url)
ters $d > 0$ and $x < 0$ correspond to the FGM/homogeneous material combinations (SiC/TiC)/TiC and (SiC/MoSi2)/MoSi2; and parameters $d < 0$ and $x > 0$ correspond to (TiC/SiC)/SiC and (MoSi2/SiC)/SiC. These regions are indicated in Figs. 6 and 7. The thermal properties of these materials are: $k_{SiC} = 60 \text{ W m}^{-1} \text{ K}^{-1}$, $k_{TiC} = 20 \text{ W m}^{-1} \text{ K}^{-1}$, $a_{SiC} = 4 \times 10^{-6} \text{ K}^{-1}$, $a_{TiC} = 7 \times 10^{-6} \text{ K}^{-1}$, $x_{MoSi2} = 5 \times 10^{-6} \text{ K}^{-1}$ [13,14]. The elastic modulus, Young’s modulus and Poisson’s ratio, of these materials are similar.

These examples of material combinations show that in the study of the influence of inhomogeneity parameters on the mechanical characteristics of FGMs with cracks the parameters $\delta$ and $\omega$ are usually not identical and it is not typical to put these parameters to the same value as it is performed in many studies (see, for example, [12]). In each case of actual FGMs the values of their inhomogeneity parameters should be estimated. Ref. [15] gives a number of examples of material properties of FGMs and on this basis different combinations of the non-homogeneity parameters, which control the variation of the heat conductivity, the shear modulus and the thermal expansion in the graded coating, were studied.

Another example of the influence of the system of cracks on the TSIF of the interface crack is given in Table 1. The TSIFs at the right interface crack tip are presented for randomly oriented system of cracks in Fig. 3d, and for comparison for the crack system shown in Fig. 3c with $\omega = 0$, $\pi/4$ and $\pi/2$. The calculations were performed for $\delta = 1$ and $\omega = 0$. We can see the difference in the TSIFs.

Table 2 presents a calculation accounting for interface crack closure for the TSIF $K_1$ for the interface crack as influenced by a crack in the FGM with midpoint coordinate $w_k = 1.0 + i 0.2$, $x_k = 1.0$, $y_k = 0.2$, inclination angles $\theta = \pi/4$ and $\pi/3$, and inhomogeneity parameters $d > 0$ and $x < 0$.
parameters $\delta = 1$ and $\omega = 0$. The non-dimensional half length $c/a_0$ of the open portion of the interface crack is also given in the table (see Fig. 2). The case 1 corresponds the TSIFs without taking into account the interface crack closure (Eq. (22)), the case 2 corresponds the TSIFs accounting for interface crack closure. The calculation was performed using Eqs. (28) and (31) and the procedure described in Section 4 is applied. The parameter $V$ (30) is equal to 1, i.e. the microcrack is open. The small crack near the right interface crack tip causes the interface crack shielding, $K_1^+$ is negative. A small contact zone near the right crack tip appears, the half length of this zone is equal to $(a_0 - c)/a_0 = 0.073$ for $\pi/4$ and 0.05 for $\pi/3$. The recalculation of the TSIFs accounting for interface crack closure gives the difference in 25% between case 1 and case 2 for $\pi/4$ and in 12.3% for $\pi/3$.

6. Conclusions

An approximate model for investigating the influence of microcracks on an interface crack in FGM/homogeneous bimaterials under a heat flux is presented and based on the asymptotic analytical formulas for the thermal stress intensity factors at the interface crack tips. The parametric analysis shows the dependence of the TSIFs at the interface crack tips on the location and orientation of the cracks in the FGM. It is also shown that the inhomogeneity parameters $\delta$ and $\omega$ of thermo-conductivity and thermal expansion coefficient notably affect the TSIFs of the interface crack. The TSIFs can be amplified or shielded by the system of microcracks. It is also shown, that for some crack arrangements (Fig. 6a) the TSIF $k_1$ at the interface crack tips can be negative, i.e. the interface crack surfaces could close. The solution taking into account the crack closure is derived and an example of this calculation is given. These results are applicable to such kinds of FGMs as: ceramic/ceramic FGMs, i.e. TiC/SiC, MoSi$_2$/Al$_2$O$_3$ and MoSi$_2$/SiC, and also some ceramic/metal FGMs, i.e. zirconia/nickel and zirconia/steel. Examples of some FGM/homogeneous bimaterials are discussed.

Acknowledgement

V. Petrova acknowledges the support of the German Research Foundation under Grants Schm 746/92-1 and Schm 746/106-1.

References